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Momentum disturbances and wave trains

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The effects of a reduction in momentum flux on the downstream flow characteristics of a steady, two-dimensional flow are investigated. In particular, quantities such as the changes in mean depth, mean fluid velocity, mean kinetic and potential energies and the length of the induced downstream wave are examined. This is done with the use of a fourth-order perturbation expansion in wave slope. The results are compared with the conflicting results that had been obtained previously by different authors. Agreement is found with the second-order theory of Benjamin (1970), but the work of Doctors & Dagan (1980) is found to be in error and is corrected.

1. Introduction

The basic concern of this paper is the effect of a stationary disturbance on the behaviour of a two-dimensional free-surface flow far downstream. The disturbance is one whereby the momentum of the fluid is changed but energy is conserved, as by a pressure patch on the surface, bottom topography or an obstacle within the fluid. This can cause a downstream wave system, which is portrayed in figure 1. The study of the properties of this wave system has had a long history but comparatively little has been said concerning the changes in quantities such as mean depth and mean fluid velocity of the downstream flow and wavenumber of the induced waves due to changes in disturbance strength.

Consider an open horizontal channel containing an incompressible, inviscid fluid of depth D, flowing irrotationally to the left with velocity U. A stationary wave system will be formed downstream from a steady disturbance if U is not too large. It is well known that a stationary wave train can be uniquely defined by Q, the volume flow rate per unit span, R, the energy per unit mass (Bernoulli's constant), and S, the momentum flow rate per unit span, corrected for changes in horizontal pressure force. By conservation of mass, Q is constant throughout the regions of study, and provided the disturbance is frictionless and the fluid irrotational R will also be constant. Thus the disturbance is equivalent to a change in the momentum flux alone. For the purpose of this paper the disturbance strength is defined as the reduction in the momentum flow rate per unit span from its upstream value i.e. S_0-S, where S_0 is the momentum flow rate upstream of the disturbance and S is its value downstream. S_0-S is also equal to the wave resistance of the disturbance.

The case when the depth is small in comparison with wavelength was investigated by Benjamin & Lighthill (1954). The authors used shallow-water theory, expressed in terms of Q, R and S alone, to give the allowable regions of R and S for a wave solution to exist. For given Q and R corresponding to a uniform subcritical flow upstream, it was found that as the disturbance strength S_0-S is increased the amplitude of

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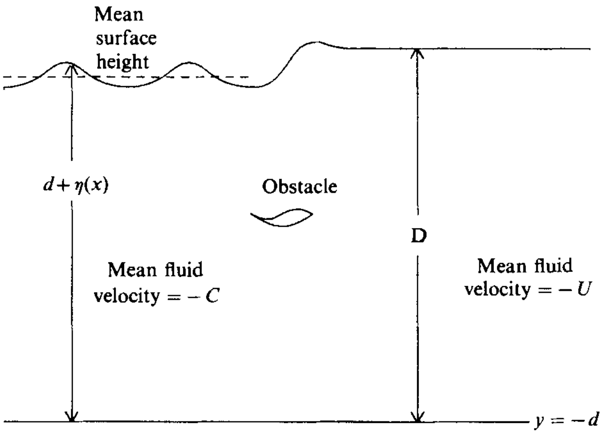

FIGURE 1. Disturbance causing changes in the flow parameters.

the waves increases and eventually, as the maximum possible value of $S_0 - S$ is approached, the wavelength diverges to infinity, leaving a uniform supercritical flow downstream.

The study of the flow when the depth is not shallow has not been as straightforward. In a paper on upstream influence, Benjamin (1970) determined a second-order expression for the change in depth between the region far upstream and the region of the stationary wavetrain. This is

$$k\hat{h} = \frac{(\frac{1}{2}kH)^2\,(1+kD\,\mathrm{cosech}\,(2kD))}{2kD(1-F^2)}, \tag{1.1}$$

where $\hat{h}$ is the change in mean depth, k is the wavenumber and H is the crest-to-trough height of the downstream wave. F is defined as $U/(gD)^{\frac{1}{2}}$, where g is the gravitational acceleration. This mean drop in surface height was confirmed experimentally by Salvesen & von Kerczek (1978).

Doctors & Dagan (1980) investigated the effect that a surface pressure disturbance has on the flow far downstream from the disturbance. A perturbation expansion was developed and expressions for the velocity potential and surface height far downstream were calculated to third order. These solutions gave no change in the mean fluid velocity between upstream and downstream flow, but a drop in the mean surface height was given, to second order in waveheight, by

$$k\hat{h} = (\tfrac{1}{2}kH)^2/2\sinh(2kD). \tag{1.2}$$

This is clearly not in agreement with (1.1).

De (1955) devised a method to define the properties of a wave system in terms of R and S, using a fifth-order Stokes-type expansion. For a given R and S the values of kH and $k\bar{h}$ were found, where $\bar{h}$ is the mean depth in the region of the wavetrain. It is important to note that this method does not give complete information about the wave system as the wavenumber itself is a function of waveheight and mean depth. Thus the quantities k, H and $\bar{h}$ cannot be found individually and the paper does not relate the downstream mean depth to that of the upstream depth. It was also claimed by Chappelear (1961) and confirmed later by Fenton (1985) that the higher-order results of De are incorrect.

Salvesen (1969) showed that for the case of infinite depth the length of the downstream waves is given by the equation

$$\lambda = \frac{2\pi U^2}{g}(1-(\tfrac{1}{2}kH)^2), \tag{1.3}$$

which is in contrast to the shallow-water results, as here the wavelength decreases with disturbance strength. The wavelength dependence was further investigated in a later paper (Salvesen & von Kerczek 1978), where Benjamin's second-order results for the change in the downstream mean depth were combined with De's perturbation results for free waves. This gave a method to calculate the wavelength for a given waveheight and Froude number. For Froude numbers less than 0.73 the wavelength was found to decrease with waveheight, but to increase with waveheight when the Froude number was larger. These results should be viewed with caution, however, as Benjamin's results have been brought into question and De's results are incorrect at high orders.

As can be seen, much of the work on this topic when the depth is not small is either incomplete, in conflict, or has been shown to be incorrect at high orders. The motivation for the present paper was (*a*) to develop a high-order method which would simultaneously investigate all the above effects and hence check validity of these various theories; (*b*) to use these higher-order corrections as an error estimate for the lower-order results; (*c*) to evaluate the integral quantities for changes in mean kinetic energy and mean potential energy caused by the disturbance, taking into account these changes in mean depth and wavelength. Previous papers, e.g. Cokelet (1977), estimated these energy changes for an increasing waveheight but with fixed wavelength. This caused R to vary with waveheight.

2. Perturbation method

Consider the flow portrayed in figure 1. At some point in the flow there is a disturbance causing a momentum loss to the fluid. If the flow is subcritical then waves will be induced downstream. The flow can be considered to have two distinct regions separated by the disturbance. The first is the steady uniform flow far enough upstream for the local effects of the disturbance to be ignored and the second, the wave system with reduced momentum flux, is far enough downstream for the local effects of the disturbance to be again ignored.† The steady and uniform upstream flow is fully determined by depth D and velocity U, or equivalently by U and $F = U/(gD)^{\frac{1}{2}}$. Thus Q, R and S for the upstream flow are given by

$$Q_0 = UD = U^3/gF^2,$$

$$R_0 = \tfrac{1}{2}U^2+g(D-d),$$

and

$$S_0 = U^2D+\tfrac{1}{2}gD^2 = U^4(1+\tfrac{1}{2}F^2)/gF^2.$$

Assuming that the downstream flow is also steady with $Q = Q_0$ and $R = R_0$ everywhere, solving for the downstream flow is reduced to finding a wave system with $Q = Q_0$, $R = R_0$ and $S = S_0 - \Delta S$, ΔS being the disturbance strength. The

† As pointed out by Benjamin (1970) the flow in this upstream region differs from the flow prior to the formation of the disturbance, having a different depth and velocity. The present upstream flow was established after upstream effects, caused by the initialization of the disturbance, had advanced sufficiently far ahead.

amplitude and other properties of the downstream periodic wave system can then be calculated as functions of $\Delta S > 0$, parameterized by $F < 1$.

The advantage of this approach, whereby the values of conserved quantities from either side of the disturbance are matched, is that the equations of motion do not have to be solved in the region of the disturbance, often a somewhat daunting task. The following results are independent of the detailed structure of the flow field near the disturbance, in that only knowledge of the reduction in S due to the disturbance is necessary.

In this section a fourth-order perturbation expansion is derived for the downstream flow, such that the Bernoulli constant for this wave system is the same as that of the upstream flow. In §3 the volume flux for this wave system is found and equated to that of the flow ahead of the disturbance. This gives a single nonlinear algebraic equation to solve as the matching condition. From this, the physically possible wave system can be selected.

The axes are defined such that positive x is horizontal and upstream while y is vertically upwards, with the bed at $y = -d$. The surface is defined at $y = \eta(x)$, with $\eta(x)$ reducing to zero with disturbance strength. The length d is not known *a priori* and is essentially a mathematical convenience, but must be D to zeroth order. This dual notation may seem confusing but is needed as the two regions are examined separately at first. By varying d the mean downstream depth is also varied. The value of d is then selected such that the volume flux downstream is Q_0.

The flow can be described by the use of a velocity potential, $\phi(x, y)$, such that

horizontal velocity, $$u(x,y) = \phi_x(x,y) - C, \tag{2.1a}$$

vertical velocity, $$v(x,y) = \phi_y(x,y), \tag{2.1b}$$

where C is the mean downstream fluid velocity so far downstream that ϕ_x contains only terms which have a zero mean over one wavelength. Incompressibility implies that

$$\nabla^2\phi = 0, \tag{2.2}$$

and if pressure is defined as being zero at the surface then

$$\tfrac{1}{2}(\phi_x - C)^2 + \tfrac{1}{2}\phi_y^2 + gy = R \quad \text{at} \quad y = \eta(x), \tag{2.3}$$

from Bernoulli's theorem. The dynamic boundary conditions give

$$(\phi_x - C)\,\eta_x - \phi_y = 0 \quad \text{at} \quad y = \eta(x), \tag{2.4}$$

and $$\phi_y = 0 \quad \text{at} \quad y = -d. \tag{2.5}$$

These equations are nonlinear and thus difficult to solve for the downstream case owing to the presence of the wave system. Using perturbation methods they can be approximated by a set of simultaneous linear equations.

Define

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \epsilon^4\phi_4 + \epsilon^5\phi_5 + \ldots, \tag{2.6a}$$

$$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + \epsilon^4\eta_4 + \epsilon^5\eta_5 + \ldots, \tag{2.6b}$$

and $$C = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + \epsilon^4 c_4 + \epsilon^5 c_5 + \ldots, \tag{2.6c}$$

were ϵ is related to the size of the disturbance.

Note that R is not expanded, unlike in De's formulation. The solution to this system is then the family of waves with the same Bernoulli constant. The above expansions can be substituted into (2.3) and (2.4). Since (2.3) and (2.4) are evaluated at the surface, the actual values needed are $\phi_{i_x}(x, \eta)$ and $\phi_{i_y}(x, \eta)$. These can be obtained to the same order of approximation by expanding in y about $y = 0$. All this was done with the use of MACSYMA, a computer algebraic manipulation package which can perform symbolic substitution, differentiation, etc. Upon the collection of terms of like order in ϵ, sets of linear equations were obtained as the boundary conditions. Owing to their length these are not published here but may be obtained from the author.

The equations for the Bernoulli condition are of the form

$$\tfrac{1}{2}c_0^2 = R \quad \text{for} \quad i = 0 \tag{2.7}$$

and
$$\phi_{i_x} - \frac{\eta_i g}{c_0} = f_i(x) + c_i \quad \text{at} \quad y = 0 \quad \text{for} \quad i > 0. \tag{2.8}$$

The surface kinematic conditons are of the form

$$\frac{\eta_{i_x}(x)\, g}{c_0} + k_0\, \phi_{i_y} = g_i(x) \quad \text{at} \quad y = 0 \quad \text{for} \quad i > 0. \tag{2.9}$$

f_i and g_i are functions of lower-order terms and

$$k_0 = \frac{g}{c_0^2}. \tag{2.10}$$

In the linear approximation, k_0 is equivalent to the infinite-depth wavenumber. For $i > 0$, (2.8) and (2.9) combine to give

$$\phi_{i_{xx}} + k_0\, \phi_{i_y} = f_{i_x}(x) + g_i(x) \quad \text{at} \quad y = 0. \tag{2.11}$$

The linear solution to the problem is then of the form

$$\phi_1(x, y) = \frac{Ua_1 \cosh(k(y+D)) \sin(kx)}{\sinh(kd)}, \tag{2.12a}$$

$$\eta_1(x, y) = a_1 \cos(kx) \tag{2.12b}$$

and
$$C = U, \tag{2.12c}$$

where k satisfies
$$kd \coth(kd) = \frac{gd}{c_0^2} = \frac{1}{F^2}. \tag{2.13}$$

a_1 is half the linear waveheight. This is simply a wave superimposed on the undisturbed steady stream. Note that the mean depth and fluid velocity are unchanged downstream at this order.

With the use of extensive programming in MACSYMA, expansions of higher orders were then evaluated for $\phi(x, y)$, $\eta(x)$ and C, by substitution of the lower-order terms back into (2.11) to obtain $\phi_i(x, y)$ and then (2.9) was evaluated to obtain $\eta_i(x)$. The values for the term c_i were found by the restriction that the term $\phi_{i+1}(x, y)$ must be bounded. Hence the right-hand side of (2.11), at this order, cannot contain a harmonic term with wavenumber k as otherwise the solution for $\phi_{i+1}(x, y)$ would contain a resonant term. In this manner a set of fourth-order solutions was obtained for the above quantities.

The wave slope kH can then be found from

$$kH = k(\eta(0) - \eta(\pi/k)). \tag{2.14}$$

This expression was used to express the fourth-order solution in terms of $\frac{1}{2}kH$ rather than a_1.

$\frac{1}{2}kH$	Bernoulli	kinematic	$\frac{1}{2}kH$	Bernoulli	kinematic
$kd = 1.4658$, $kx = 1.2454$			$kd = 2.12$, $kx = 0.45$		
0.157			0.11		
	5.27	4.42		0.93	5.06
0.11			0.076		
	5.18	4.81		4.00	5.04
0.076			0.049		
	5.11	4.92		4.59	5.03
0.049			0.029		
	5.07	4.98		4.80	5.02
0.029			0.017		
	5.02	4.99		4.91	5.01
0.017			0.0077		
	5.01	5.00		4.96	5.00
0.0077			0.0027		
	5.07	5.00			
0.0027					

TABLE 1. Exponents of error for surface boundary conditions

These solutions for $\phi(x, y)$, $\eta(x)$ and C were checked by substitution back into the original boundary conditions, (2.3) and (2.4). For a given $\frac{1}{2}kH$, kd and kx a value for surface height $k\eta(x)$ can be determined. Thus $\phi_x(x, \eta)$ and $\phi_y(x, \eta)$ can be found and used to evaluate (2.3) and (2.4). The error in both equations can then be determined for various values of $\frac{1}{2}kH$, A MACSYMA program was written to perform these analytic and numeric operations successively. It was assumed that the error was of the form $(\frac{1}{2}kH)^j$, with j being found from neighbouring values of $\frac{1}{2}kH$ and their corresponding errors. The resulting values for j can be found in table 1. It can be seen that as $\frac{1}{2}kH$ reduced to zero, the error behaves as $(\frac{1}{2}kH)^5$. This will only occur if the solutions are correct to fourth order.

3. Derivation of integral quantities and formulae for changes in flow parameters

Now that $\phi(x, y)$, $\eta(x)$ and C are known to fourth order, similar order expressions are possible for the volume flux

$$Q = \int_{-d}^{\eta} u(x, y)\,\mathrm{d}y, \tag{3.1}$$

for the horizontal momentum flux, corrected for pressure and divided by density,

$$S = \int_{-d}^{\eta} \left(\frac{p}{\rho} + u^2\right) \mathrm{d}y, \tag{3.2}$$

where the term p/ρ can be found from Bernoulli's equation; and for the mean potential and kinetic energy, divided by density and averaged over one wavelength,

$$V = \frac{1}{\lambda} \int_{x_0}^{x_0+\lambda} \int_{-d}^{\eta} gy\,\mathrm{d}y\,\mathrm{d}x, \tag{3.3}$$

and

$$T = \frac{1}{2\lambda} \int_{x_0}^{x_0+\lambda} \int_{-d}^{\eta} (u^2(x, y) + v^2(x, y))\,\mathrm{d}y\,\mathrm{d}x, \tag{3.4}$$

respectively. The vertical domain of integration for all these quantities is from the bed to the surface of the flow $y = \eta$, where $\eta = (D-d)$ far upstream.

These integrals were evaluated for the downstream case using the solutions for $\phi(x, y)$, $\eta(x)$ and C from §2. As $\eta(x)$ is 'small', any function with it as an argument after integration was expanded to the correct order. In this manner, fourth-order solutions for these integral quantities were obtained in terms of c_0, kd and $\frac{1}{2}kH$. The far downstream expressions for Q and S passed the stringent test of being independent of x, which is expected as there is no external source of fluid or forcing term.

These downstream solutions were non-dimensionalized so that

$$Q/c_0 d = \hat{q}(\tfrac{1}{2}kH, kd), \quad S/c_0^2 d = \hat{s}(\tfrac{1}{2}kH, kd), \tag{3.5a,b}$$

$$V/c_0^2 d = \hat{v}(\tfrac{1}{2}kH, kd), \quad \text{and} \quad T/c_0^2 d = \hat{t}(\tfrac{1}{2}kH, kd), \tag{3.5c,d}$$

where $\hat{q}$, $\hat{s}$, $\hat{v}$ and $\hat{t}$ can be found in Appendix A.

Equating the values for the upstream and downstream Bernoulli constant and volume flux, one obtains using (2.7)

$$R = \tfrac{1}{2}c_0^2 = \tfrac{1}{2}U^2 + g(D-d), \tag{3.6}$$

and
$$\hat{q}(\tfrac{1}{2}kH, kd)\,(F^2+2)^{\frac{3}{2}}\,kd \coth(kd) + F(1+2kd\coth(kd))^{\frac{3}{2}} = 0, \tag{3.7}$$

from (3.1), (3.5), (3.6) and (2.13). By solving (3.7) for a given wave slope, one obtains the value of kd for which the volume flux is consistent between the two regions. Hence all the above terms can then be evaluated for a given wave slope. It should be stressed that the term kd has no simple physical meaning but is a mathematical artefact dependent upon the wave slope and Froude number.

The change in mean horizontal speed is given by

$$\frac{U-C}{U} = 1 - \frac{(F^2+2)^{\frac{1}{2}}\,\hat{c}(\tfrac{1}{2}kH, kd)}{F(1+2kd\coth(kd))^{\frac{1}{2}}}, \tag{3.8}$$

where $\hat{c}$ is C/c_0.

The wavelength, non-dimensionalized by dividing by the linear infinite-depth wavelength, is

$$\frac{\lambda}{2\pi F^2 D} = \frac{(F^2+2)\coth(kd)}{F^2(1+2kd\coth(kd))}. \tag{3.9}$$

Similarly, the change in mean surface level is

$$\frac{\hat{h}}{F^2 D} = \frac{D-(d+\bar{\eta}(x))}{F^2 D} = \left(1 - \frac{\hat{\eta}(\tfrac{1}{2}kH, kd)\,kd\coth(kd)\,(F^2+2)}{1+2kd\coth(kd)}\right)\Big/F^2, \tag{3.10}$$

where $\hat{\eta}(\frac{1}{2}kH, kd)$ is defined as $(1 + k\bar{\eta}/kd)$, the bar indicating that it has been averaged over one wavelength.

The momentum flux was made non-dimensional by dividing by the momentum of a steady stream with horizontal velocity U and a depth equal to $1/k_0$. This gave

$$\frac{\Delta S}{\frac{3}{2}U^2F^2D} = \frac{\Delta S}{S*} = \frac{2}{3F^3}\left[\frac{2F^2+1}{2F} + \frac{\hat{s}}{q}\frac{(F^2+2)^{\frac{1}{2}}}{(1+2kd\coth(kd))^{\frac{1}{2}}}\right], \tag{3.11}$$

where ΔS is the difference between the upstream and downstream momentum flux.

Similarly the expressions for the difference in mean potential and kinetic energies become

$$\frac{\Delta V}{T^*} = \frac{(k\bar{\eta})^2 (F^2+2)^2 \coth^2(kd) - (1-F^2 kd \coth(kd))^2}{F^4(1+2kd\coth(kd))^2} \tag{3.12}$$

and

$$\frac{\Delta T}{T^*} = \frac{2\hat{t}(\tfrac{1}{2}kH, kd)\,(1+\tfrac{1}{2}F^2)^2\, kd \coth kd)}{F^4(kd\coth(kd)+\tfrac{1}{2})^2} - \frac{1}{F^2}, \tag{3.13}$$

where $T^* = \frac{1}{2}U^2F^2D$.

It can be shown that all these identities, representing changes from the upstream flow due to the disturbance, reduce to zero in the limit that $\frac{1}{2}kH \rightarrow 0$. Equation (3.9) also reduces to (2.13).

4. Results

Equation (3.7) was solved numerically for kd, given the waveheight and Froude number. It was found that for Froude numbers greater than about 0.8 there were two solutions for a given wave slope, up to a maximum value of kH when the two solutions coalesce. After this value no solution appeared to exist. For a Froude number of 0.8 this maximum kH was found to be 0.2, while for a Froude number of 0.9 it was 0.06. Physically, the two solutions correspond to different waves with the same slope. The first is a wave of small waveheight, while the second is a wave of larger waveheight but smaller wavenumber. The latter solution is associated with the larger disturbance.

Once the value of kd was found, the evaluation of the terms in Appendix A was straightforward. Convergence was found to be slow for many of the quantities when the Froude number was small. This was improved when analytic solutions were used based on an approximation for small Froude number, rather than on solving (3.7) numerically. These are derived in Appendix B and were used for Froude numbers less than 0.4. In contrast, convergence was faster than expected for Froude numbers near unity, despite the small parameter for shallow depths, $kH/(\mathrm{kd})^3$, becoming singular as the Froude number tends to unity. One reason for this would be because kH can never be large in this region, as discussed above.

The quantities defined in (3.8)–(3.13) were then plotted against changes in momentum flux. In all case the quantities were non-dimensionalized by dividing by a depth-independent quantity. In this way a change in the depth, effected by a change in the Froude number for a given U and g, does not affect the normalizing scale. The same quantity can therefore be compared for various depths. The second-order solutions were plotted as dashed curves while the fourth-order solutions are continuous lines. In the case when (3.7) had two solution branches, the solution of the first branch was initially used with increasing kH until the maximum wave slope was reached, after which the solution branch was changed and the value of kH was reduced.

The waveheight for a given disturbance strength is shown in figure 2. This was found with the use of (3.9), for a given wave slope. The trend is that as F increases, implying a decrease in depth, the same disturbance strength will cause a wave of greater height.

Figure 3 shows the increase in mean fluid velocity. Note that this is clearly non-zero. Again the effect is much greater for Froude numbers near unity where the curves are concave up, implying that each increment in disturbance strength causes

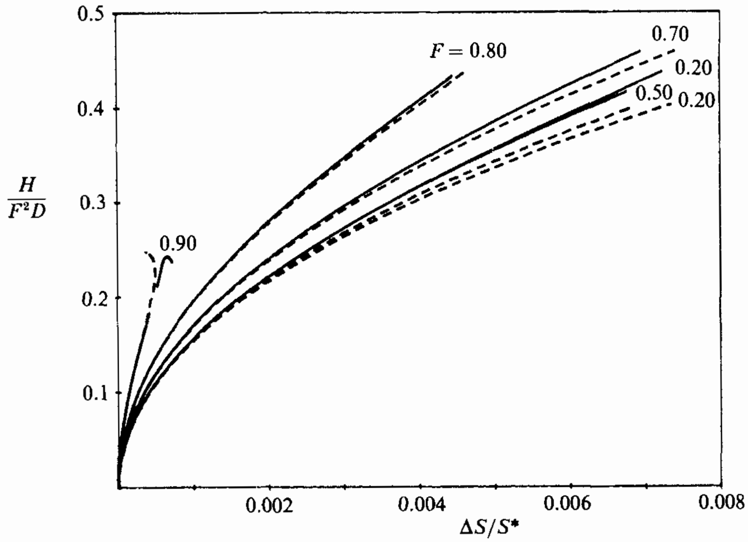

FIGURE 2. Waveheight versus reduction in momentum flux for various values of the Froude number. Second-order (– – –) and fourth order (——) results are shown.

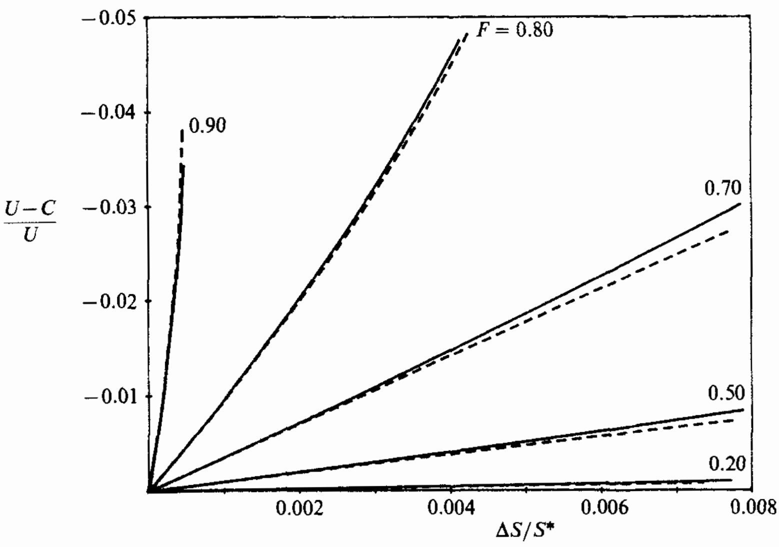

FIGURE 3. Change in mean velocity versus reduction in momentum flux.

an ever increasing change in the mean speed. The drop in mean surface height is shown in figure 4 and is nearly identical in form with the change in mean fluid velocity. Again convergence is quite acceptable, even for large Froude numbers.

The effect of the disturbance strength on the wavelength is shown in figure 5. For values of F less than 0.7 there is a decrease in the wavelength. For values of F greater than this the wavelength increases. Salvesen & von Kerczek (1978) found this critical value for Froude number to be 0.73, approximately. The discrepancy could at least partially be due to the Froude number defined by Salvesen & von Kerczek being defined in terms of the initial fluid velocity and depth, before the disturbance was generated. Also, it is to be remembered that these results are based on those of De,

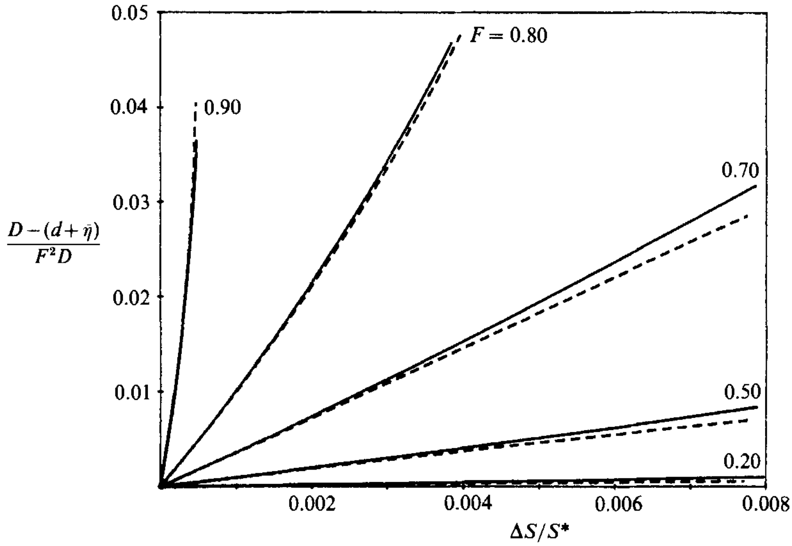

FIGURE 4. Change in mean surface height versus reduction in momentum flux.

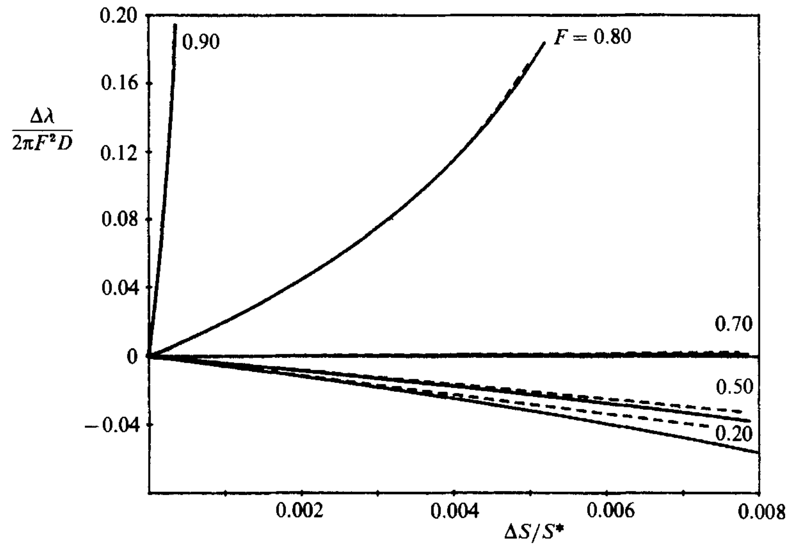

FIGURE 5. Change in wavelength versus reduction in momentum flux.

which are in error at high orders. It is assumed that the upper bound in the wave slope is caused by this increase in the wavelength for Froude numbers near unity. It is also tempting to speculate that the reason for the wavelength increase is the increase in the mean fluid velocity and the decrease in mean depth, both of which would increase the wavelength. The changes in mean kinetic energy and mean potential energy are plotted in figures 6 and 7.

It is interesting to note that to leading order all the graphed quantities, except for the waveheight, should have a linear relation with the disturbance strength. All curvature in the graphs is then attributable to higher-order effects. This implies that, especially for Froude numbers near unity, the higher orders tend to emphasize the existing trends.

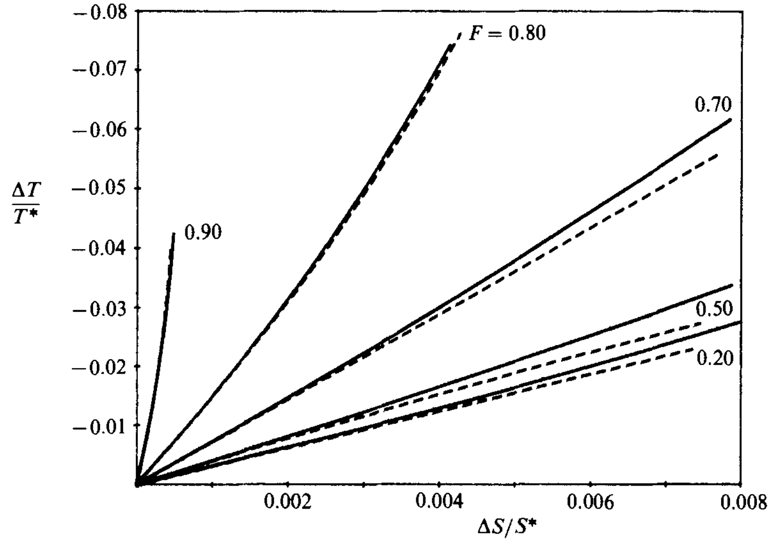

FIGURE 6. Change in mean kinetic energy versus reduction in momentum flux.

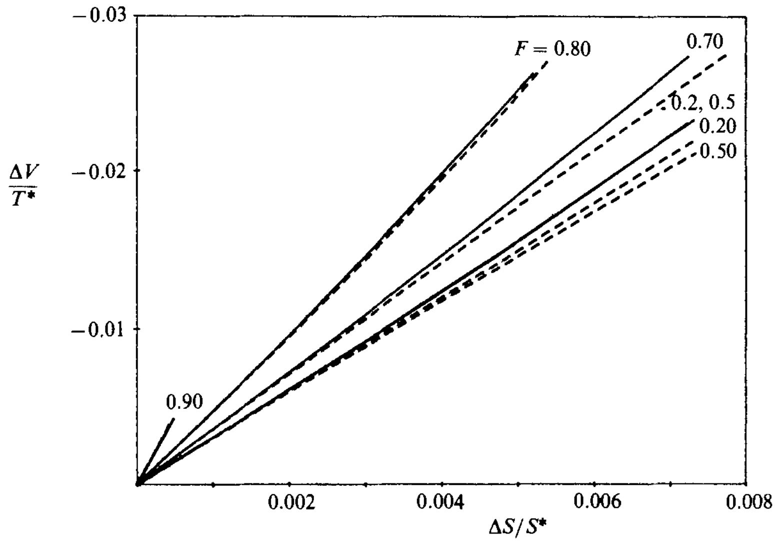

FIGURE 7. Change in mean potential energy versus reduction in momentum flux.

5. Comparison with previous results

The second-order results of the present method were compared with the results of Benjamin (1970) for the change in mean depth. The difference between (1.1) and the second-order truncated form of (3.10) was investigated. It was assumed that the difference is of the form $(\frac{1}{2}kH)^j$, where j was found by the same method as in §2. In all cases j converged to 4.00, implying that the two expressions agree to third order, for all Froude numbers. It does seem therefore that the present analysis supports the results of Benjamin concerning the decrease in mean depth.

The results for the change in mean fluid velocity are in conflict with those of Doctors & Dagan (1980). It seems necessary to account for the inconsistency.

The formulation of the problem by Doctors & Dagan was essentially the same as that of the author's, except that the upstream velocity was expanded in powers of the small parameter rather than the mean downstream velocity. The velocity potential $\hat{\phi}(x,y)$ was defined such that the local horizontal fluid velocity was given by

$$u(x,y) = \hat{\phi}_x(x,y) - U.$$

Hence, any change in the horizontal mean fluid velocity must be part of the solution for $\hat{\phi}(x,y)$.

The two surface boundary conditions, at the jth order of the solution, could be reduced to a form similar to (2.11). That is,

$$\hat{\phi}_{j_{xx}}(x,y) + k_0 \hat{\phi}_{j_y}(x,y) = h_{j_x}(x) \quad \text{at} \quad y = 0, \tag{5.1}$$

k_0 being defined here as g/U_0^2, U_0 being the zeroth-order term for U and $h_j(x)$ depending on lower-order terms.

The set of equations was solved by assuming the solution to the jth-order velocity potential to take the form

$$\hat{\phi}_j(x,y) = \mathrm{Re}\left[\int_{-\infty}^{\infty} A_j(k) \cosh\left(k(y+D)\right) \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k\right], \tag{5.2}$$

where Re designates the real part of the integral. $A_j(k)$ could be found by substituting into (5.1) and with the use of the Fourier inversion theorem.

This gave

$$\hat{\phi}_j(x,y) = \mathrm{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-(\hat{c}_j(k,x) + \mathrm{i}\hat{s}_j(k,x)) \cosh\left(k(y+D)\right)}{k \cosh(kD)\, q(k)} \mathrm{d}k\right], \tag{5.3}$$

where
$$\hat{c}_j(k,x) + \mathrm{i}\hat{s}_j(k,x) = \int_{-\infty}^{\infty} h_{j_s}(s)\, \mathrm{e}^{\mathrm{i}k(x-s)}\, \mathrm{d}s, \tag{5.4}$$

and
$$q(k) = k - k_0 \tanh kD.$$

If it is assumed that

$$\int_{-\infty}^{\infty} h_{j_s}(s)\, \mathrm{e}^{-\mathrm{i}ks}\, \mathrm{d}s = h_j(s)\, \mathrm{e}^{-\mathrm{i}ks}\Big|_{-\infty}^{\infty} + \mathrm{i}k \int_{-\infty}^{\infty} h_j(s)\, \mathrm{e}^{-\mathrm{i}ks}\, \mathrm{d}s$$
$$= \mathrm{i}k \int_{-\infty}^{\infty} h_j(s)\, \mathrm{e}^{-\mathrm{i}ks}\, \mathrm{d}s,$$

then (5.3) reduces to the expression given by Doctors & Dagan. This is true only if $|h_j(s)|$ decays to zero for large values of $|s|$. It can be shown that this will only occur for j unity, owing to the non-decaying downstream wave. There is therefore a double pole at $k = 0$, for $j > 1$, in the integrand of (5.3) which was neglected by Doctors & Dagan. Using complex residue theory, it can be shown that this has the effect of adding an extra term, linear in x, to the downstream solution for $\hat{\phi}(x,y)$. This extra term is of the form

$$\frac{(\frac{1}{2}kH)^j\, x}{(k_0 D - 1)} \lim_{k\to 0} \left(\hat{c}_j(k,x) + \mathrm{i}\hat{s}_j(k,x)\right). \tag{5.5}$$

With these corrections, the calculations of Doctors & Dagan were repeated, to second order, for the far downstream case. This now revealed a change in the mean current, given by

$$\frac{U-C}{U}=\frac{\vec{\phi}_x}{U}=-\frac{(\frac{1}{2}kH)^2\,(1+\frac{3}{4}\operatorname{cosech}^2(kD))\,F^2}{2(1-F^2)}. \tag{5.6}$$

The subsequent expression for the change in the mean depth was found to be identical to (1.1). The three methods are not consistent to second order.

6. Conclusion

The present results confirm the second-order expression for the change in mean depth derived by Benjamin (1970), caused by reduction of the momentum flux. The corresponding expression derived by Doctors & Dagan (1980) has been shown to be in error. The cause of this error has been investigated and corrected, such that all three methods are now consistent. This implies that the finite-depth results for changes of wavelength with disturbance strength, obtained by Salvesen & von Kerczek (1978), should also be correct to second order. Indeed, the present analysis shows qualitative agreement.

Another result that emerges from this analysis is that for Froude numbers above 0.7, linear analysis is quite lacking. The change in mean fluid velocity and subsequent drop in mean surface height cannot be disregarded in any analysis of a disturbance in a fluid of moderate or small depth.

The convergence of the expansions is better than expected for Froude numbers near unity. This is mainly because the wave slope is bounded from above in this region an in fact decreases with increasing disturbance strength after this upper bound is reached.

For small Froude numbers little extra information was gained from the fourth-order corrections but again, for larger Froude numbers, all derived quantities grew at an increasing rate with the disturbance strength, greater than the second-order theory predicted. Thus, these higher-order terms seem to emphasize, rather than moderate, the existing trends from lower orders.

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Appendix A. Fourth-order solution to perturbation expansion in wave slope; $(\frac{1}{2}kH)$

Note that s and c are defined such that;

$$s=\sinh kd, \quad c=\cosh kd$$

Mean surface height $k(d+\bar{\eta})$

0th order: kd;

1st order: 0;

2nd order: $-\dfrac{c(8s^4+12s^2+9)}{8(2c^2+1)\,s^3}$;

3rd order: 0;

4th order: $-c(4s^2+3)\,(256s^{12}+1792s^{10}+4224s^8+2304s^6-3348s^4$
$-4212s^2-1215)/(512s^9(2s^2+3)^3)$.

Mean fluid velocity c/c_0

0th order: 1;

1st order: 0;

2nd order: $\dfrac{(8s^6+16s^4+15s^2+9)}{8(2c^2+1)\,s^4}$;

3rd order: 0;

4th order:

$$(512s^{16}+6144s^{14}+26\,752s^{12}+50\,880s^{10}+36\,360s^8 -14\,472s^6-38\,556s^4-21\,141s^2-3645)/(512s^{10}(2s^2+3)^3).$$

Fourth-order integral quantities

Dimensionless mass flux $Q/c_0 d$

0th order in $\frac{1}{2}kH$: -1;

1st order: 0;

2nd order: $\dfrac{c(4s^2+3)^2}{8kds^3(2s^2+3)}-\dfrac{8s^6+16s^4+15s^2+9}{8s^4(2s^2+3)}$;

3rd order: 0;

4th order:

$$c(4s^2+3)\,(256s^{12}+1408s^0+2880s^8+720s^6-3996s^4 -4212s^2-1215)/(512kds^9(2s^2+3)^3)$$
$$-(512s^{16}+6144s^{14}+26\,752s^{12}+50\,880s^{10}+36\,360s^8-14\,472s^6 -38\,556s^4-21\,141s^2-3645)/(512s^{10}(2s^2+3)^3).$$

Dimensionless momentum flux $S/c_0^2 d$

0th order: $\dfrac{ckd}{2s}+1$;

1st order: 0;

2nd order: $\dfrac{(s^2+1)\,(8s^4+12s^2+9)}{8s^4(2s^2+3)}-\dfrac{c(20s^4+30s^2+9)}{8kds^3(2s^2+3)}$;

3rd order: 0;

4th order:

$$c^2(4s^2+3)\,(256s^{12}+1792s^{10}+4224s^8+2304s^6-3348s^4 -4212s^2-1215)/(512s^{10}(2s^2+3)^3)$$
$$-c(4s^2+3)^2\,(64s^{10}+256s^8+360s^6-288s^4 -864s^2-405)/(512kds^9(2s^2+3)^3).$$

Dimensionless mean kinetic energy $T/c_0^2 d$

0th order: $\frac{1}{2}$;

1st order: 0;

2nd order: $$\frac{8s^6+16s^4+15s^2+9}{8s^4(2s^2+3)}-\frac{c(4s^2+3)^2}{16kds^3(2s^2+3)};$$

3rd order: 0;

4th order:
$$\begin{aligned}&(4s^2+3)(256s^{14}+2048s^{12}+6912s^{10}+10\,272s^8+4284s^6\\&-4752s^4-5103s^2-1215)/(512s^{10}(2s^2+3)^3)\\&-c(4s^2+3)(768s^{12}+3584s^{10}+6720s^8+4608s^6-1620s^4\\&-3564s^2-1215)/(1024kds^9(2s^2+3)^3).\end{aligned}$$

Dimensionless potential energy $V/c_0^2 d$

0th order: $-\dfrac{ckd}{2s}$;

1st order: 0;

2nd order: $\dfrac{c}{4kds}$;

3rd order: 0;

4th order: $$-\frac{3c(4s^2+3)(8s^4+16s^2+9)}{128kds^5(2s^2+3)^2}.$$

Appendix B. Evaluation of flow characteristics for small Froude numbers

It is possible for simple analytic expressions to be found for the quantities defined in §3, at low Froude numbers, as these solutions can be greatly simplified by the approximation that $\tanh kd$ is unity.

First, a value for kd is needed in terms of F and $\frac{1}{2}kH$. This is found by assuming the form

$$kd = A+B(\tfrac{1}{2}kH)+C(\tfrac{1}{2}kH)^2+D(\tfrac{1}{2}kH)^3+E(\tfrac{1}{2}kH)^4+\dots$$

and substituting into the low-Froude number approximation of (3.7). After expanding in $\frac{1}{2}kH$ this gave

$$\begin{aligned}kD = \frac{1}{F^2}+(\tfrac{1}{2}kH)^2\frac{(1-2F^2)(2+F^2)}{2F^2(1-F^2)}\\+(\tfrac{1}{2}kH)^4\frac{(1-2F^2)(2+F^2)(2+2F^2-F^4)}{8F^2(1-F^2)^3}+\dots\end{aligned} \quad (1)$$

All other quantities can then be evaluated for a given wave slope.

From (3.8)–(3.13):

$$\frac{U-C}{U} = -(\tfrac{1}{2}kH)^2\frac{F^2}{2(1-F^2)}+(\tfrac{1}{2}kH)^4\frac{F^2(8-22F^2+11F^4)}{8(1-F^2)^3}+\dots, \tag{B 2}$$

$$\frac{\lambda}{2\pi F^2 D} = 1-(\tfrac{1}{2}kH)^2\frac{(1-2F^2)}{(1-F^2)}+(\tfrac{1}{2}kH)^4\frac{(1-2F^2)(2-14F^2+9F^4)}{4(1-F^2)^3}+\dots, \tag{B 3}$$

$$\frac{\hat{h}}{F^2D} = (\tfrac{1}{2}kH)^2\frac{F^2}{2(1-F^2)}-(\tfrac{1}{2}kH)^4\frac{(8-23F^2+12F^4)F^2}{8(1-F^2)^3}+\dots, \tag{B 4}$$

$$\frac{\Delta S}{S*} = \frac{(\tfrac{1}{2}kH)^2}{6}-(\tfrac{1}{2}kH)^4\frac{7-12F^2}{12(1-F^2)}+\dots, \tag{B 5}$$

$$\frac{\Delta V}{T*} = -\frac{(\tfrac{1}{2}kH)^2}{2}+(\tfrac{1}{2}kH)^4\frac{(1-2F^2)(5-6F^2)}{4(1-F^2)^2}+\dots, \tag{B 6}$$

and

$$\frac{\Delta T}{T*} = -\frac{(\tfrac{1}{2}kH)^2}{2(1-F^2)}+(\tfrac{1}{2}kH)^4\frac{(8-22F^2+11F^4)}{8(1-F^2)^3}+\dots. \tag{B 7}$$

It can be seen that both the change in mean depth and mean fluid velocity behave as F^2, for small F. This implies that these quantities decrease as $1/D$, for large depths. When F equals zero the above expression for wavelength, truncated to second order, reduces to (1.3).

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